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Regularization of Legendre Function Series for Charged Particles Improved Nearside-Farside Subamplitudes

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A simple regularization procedure is proposed for the Legendre function series of improved nearside-farside subamplitudes for charged particles elastic scattering. The procedure is the extension of the usual one which defines the partial wave series for the scattering amplitude in the presence of a long range Coulomb term in the potential, and it provides the same convergence rate.

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The nearside-farside (NF) method proposed by Fuller [1] is an effective tool to separate the full elastic scattering amplitude $f(\theta)$, where θ is the scattering angle, into simpler subamplitudes [2, 3]. The Fuller NF subamplitudes are usually more slowly varying and less structured then $f(\theta)$. This allows one to explain the complicated patterns appearing in some cross sections, $\sigma(\theta) = |f(\theta)|^2$, as interference effects between simpler nearside (N) and farside (F) subamplitudes. These subamplitudes can often be interpreted as contributions from simple scattering mechanisms allowing a physical understanding of the scattering process [2].

Sometimes, particularly when applied to scattering of α particles and light heavy-ions at intermediate and high energies, the Fuller NF subamplitudes are biased by the presence of unphysical contributions, making the NF subamplitudes more structured then desired. Recently an improved NF method has been proposed [4, 5] to further extend the effectiveness of the original Fuller technique. The improved NF method is based on a modified [6] Yennie, Ravenall, and Wilson (YRW) [7] resummation identity, which holds for Legendre polynomial series (LPS). The increased effectiveness descends from using resummation parameters with values reducing the unphysical contributions to the Fuller NF subamplitudes.

The Legendre function series (LFS) for the improved NF subamplitudes are, however, not convergent in the usual sense. A resummation technique [8], named in the following extended YRW (EYRW) resummation, was used in Refs. [4, 5] to obtain convergent series. At forward angles, the rate of convergence of the EYRW series is not satisfactory in the presence of a long range Coulomb term in the potential. For α particles, light and heavy ions scattering this fact is disturbing, because it compels one to use more partial waves then necessary in standard optical potentials calculations and in the usual Fuller NF method. Here we present a regularization procedure that, if applied to LFS of improved NF subamplitudes, makes these series as rapidly convergent as those of more conventional approaches.

The starting point for the improved NF method is the quantum mechanical partial wave series (PWS) of the elastic scattering amplitude

$$f(\theta) = \sum_{l=0}^{\infty} a_l P_l(\cos\theta), \tag{1}$$

where $x = \cos\theta$, $P_l(x)$ is the Legendre polynomial of degree l, $x \neq 1$, and a_l is given in terms of the scattering matrix element S_l by

$$a_l = \frac{1}{2ik}(2l+1)S_l, \tag{2}$$

where k is the wavenumber.

To obtain the improved NF subamplitudes, one substitutes the usual factor $S_l - 1$ with S_l on the r.h.s. of (2). The dropped term ensured the convergence of (1) for scattering by short range potentials, for which $S_l \to 1$ exponentially for $l \to \infty$ ([9], p. 82). In this case, having omitted a term $\propto \delta(1-x)$, where δ indicates the Dirac distribution (e.g. see [10], p. 52), the sum in (1) is defined only in a distributional sense. In the presence of a long range Coulomb term in the potential the dropped 1 is not relevant for convergence. With or without the 1, the sum in (1) is convergent only in a distributional sense. In [11, 12, 13, 14, 15], and in references therein, one can find more or less recent discussions on the convergence of the Coulombic PWS, and of the different techniques (Padè approximants, Abel summation, or different regularization procedures) solving the problem.

The improved NF subamplitudes are obtained by using for $f(\theta)$, in place of (1), its resummed form

$$f(\theta) = \left(\prod_{i=0}^{r} \frac{1}{1+\beta_i x}\right) \sum_{n=0}^{\infty} \alpha_n^{(r)} P_n(x), \tag{3}$$

$r = 0, 1, 2, \ldots$, where

$$\alpha_n^{(i)} = \beta_i \frac{n}{2n-1}\alpha_{n-1}^{(i-1)} + \alpha_n^{(i-1)} + \beta_i \frac{n+1}{2n+3}\alpha_{n+1}^{(i-1)}, \tag{4}$$

with $\beta_0 = 0$, $\alpha_n^{(0)} = a_n$, and $\alpha_{-1}^{(i)} = 0$. The resummed form (3) is an *exact mathematical identity* deriving from the recurrence property of the Legendre polynomials. It holds for real or complex values of the resummation parameters β_i $(i \neq 0)$, restricted only by the condition

$1+\beta_i x \neq 0$, for $-1 \leq x < 1$. The integer index r is the order of the resummation, and $r=0$ means no resummation of the original PWS. In (3) we changed the index of the sum (1) (from l to n) to remark that the index of the resummed Legendre polynomial series (LPS) in (3) has not, for $r \neq 0$, the physical meaning of orbital quantum number, differently from the index of the original PWS (1). Similarly the terms $\alpha_n^{(i)}$ have not the physical meaning of partial wave amplitudes. The usual YRW resummed form [7] for $f(\theta)$ is obtained by setting $\beta_i = -1$ ($i \neq 0$) in (3).

We note that for pure Coulomb scattering, for which

$$\alpha_n^{C(0)} \equiv a_n^C = \frac{1}{2ik}(2n+1)\frac{\Gamma(n+1+i\eta)}{\Gamma(n+1-i\eta)} \tag{5}$$

where η is the Sommerfeld parameter, by using (4) one obtains, for large n values,

$$\alpha_n^{C(1)} = [1+\beta_1+O(n^{-2})]\alpha_n^{C(0)}. \tag{6}$$

This means that for $\beta_i \neq -1$ the asymptotic Coulombic behavior of $\alpha_n^{C(r)}$ does not depend, apart from a renormalization factor, on the resummation order r. On the other hand, given a resummed LPS of order r (eventually 0), by applying an additional YRW ($\beta_{r+1}=-1$) resummation one obtains a convergent series for asymptotically Coulombic $\alpha_n^{(r)}$. Any successive YRW resummation improves the LPS convergence by a factor $O(n^{-2})$.

The improved NF subamplitudes are obtained by splitting in (3) the $P_n(x)$ into traveling angular components

$$P_n(x) = Q_n^{(-)}(x) + Q_n^{(+)}(x), \tag{7}$$

where (for $x \neq \pm 1$)

$$Q_n^{(\mp)}(x) = \frac{1}{2}[P_n(x) \pm \frac{2i}{\pi}Q_n(x)], \tag{8}$$

with $Q_n(x)$ the Legendre function of the second kind of degree n. By inserting (7) into (3), $f(\theta)$ is separated into the sum of two subamplitudes

$$f(\theta) = f_{\{\beta\}}^{(-)}(\theta) + f_{\{\beta\}}^{(+)}(\theta), \tag{9}$$

with

$$f_{\{\beta\}}^{(\mp)}(\theta) = \left(\prod_{i=0}^{r}\frac{1}{1+\beta_i x}\right)\sum_{n=0}^{\infty}\alpha_n^{(r)}Q_n^{(\mp)}(x). \tag{10}$$

In (10), with the subscript $\{\beta\}$ we indicate that the N ($f_{\{\beta\}}^{(-)}$) and F ($f_{\{\beta\}}^{(+)}$) subamplitudes depend, differently from $f(\theta)$, on the resummation order r and parameters β_i. This occurs because the resummed form of series (LFS) of linear combination of first and second kind Legendre functions, of integer degree, is different from (3). In fact, let us indicate with

$$\mathcal{F}(\theta) = \sum_{n=0}^{\infty} d_n \mathcal{L}_n(x) \tag{11}$$

a LFS in $\mathcal{L}_n(x) = pP_n(x) + qQ_n(x)$, with p and q independent of n. Owing to the property $nQ_{n-1}(x) \to 1$ as $n \to 0$ [8], the resummed form of $\mathcal{F}(x)$, of order s and parameters γ_i, is [5]

$$\begin{aligned}\mathcal{F}(\theta) \ &= \ \left(\prod_{i=0}^{s}\frac{1}{1+\gamma_i x}\right)\sum_{n=0}^{\infty}\delta_n^{(s)}\mathcal{L}_n(x) \\ &+ \ q\sum_{i=0}^{s}\gamma_i\delta_0^{(i-1)}\prod_{j=0}^{i}\frac{1}{1+\gamma_j x}.\end{aligned} \tag{12}$$

Equation (12) is an *exact mathematical identity* extending the validity of (3) to more general LFS, and it reduces to (3) for LPS ($q=0$). The conditions of validity of (12), and the recurrence relation for the resummed coefficients, are the same as those for (3), after substituting r, β, α, and a with s, γ, δ, and d, respectively.

Because the $Q_n^{(\mp)}(x)$ used to split $P_n(x)$ in (7) are a particular case of the more general $\mathcal{L}_n(x)$ (with $p=1/2$, and $q=\pm i/\pi$), the presence of the last term in (12) is responsible for the dependence of $f_{\{\beta\}}^{(\mp)}(\theta)$ on r and β_i. The last term on the r.h.s. of (12) gives a contribution if the splitting (7) is inserted in (1). This contribution is absent if the splitting is inserted in (3).

In Refs. [4, 5] it was observed that unphysical contributions, when appearing in the Fuller NF subamplitudes ($r=0$ in (10)), decrease by increasing r in (10) (the values $r=1$, and 2 were tested), if the β_i are selected to make null the coefficients $\alpha_0^{(r)}, \alpha_1^{(r)}, \ldots \alpha_{r-1}^{(r)}$ of the resummed LFS ($\alpha_0^{(1)}$, and $\alpha_{0,1}^{(2)}$ for the cases tested). In this way one drops the contributions to the NF resummed subamplitudes from low n values for which the splitting (7), though exact by construction, is not expected to be physically meaningful.

The $\alpha_n^{(r)}$ in (3) and (10) go asymptotically to a constant for short range potentials, or are Coulombic in the presence of a Coulomb term in the potential. Because of this the corresponding LFS are not convergent in the usual sense. In Refs. [4, 5] the convergence was forced, and accelerated, by applying to the improved LFS a final (EYRW) resummation (12), of order $s \geq 1$, with $d_n = \alpha_n^{(r)}$, $\gamma_i = -1$, and $i \neq 0$. The final EYRW resummation ensures the numerical convergence of the LFS, with a convergence rate increasing with s. The increased rate of convergence costs, however, the cancellation of significant digits (see [8] for details), and numerically the procedure may results not convenient or even impossible, using arithmetic with a fixed digit number.

These troubles can be avoided by investigating the properties of the resummation identity (12) with d_n equal to the pure Coulomb a_n^C given by (5). In this case we explicitly know the l.h.s. of (12) for the relevant p and q values. In fact, if $p=1$ and $q=0$ it is the Rutherford scattering amplitude $f_R(\theta)$, while for $p=1/2$ and $q=\pm i/\pi$ one obtains the Fuller-Rutherford NF subamplitudes $f_{FR}^{(\mp)}(\theta)$ ([1] Eqs. 14 a, b). Because (12) is exact

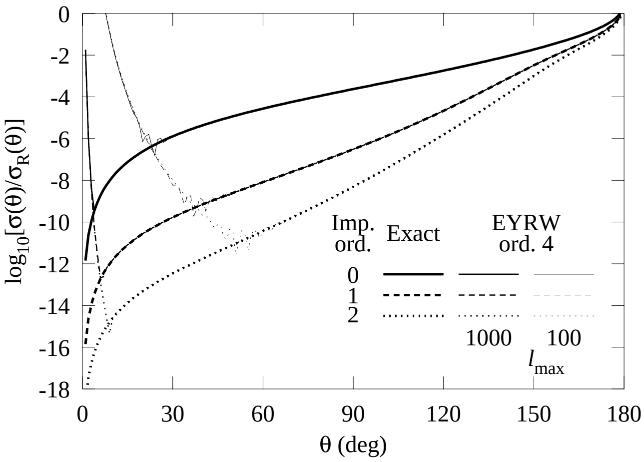

FIG. 1: Different order ($r = 0, 1$ and 2) improved F pure Coulomb cross sections calculated with the exact expression (thick curves) and using an EYRW resummation of order 4, with 1000 (medium thickness curves) and 100 (thin curves) partial waves.

it holds for arbitrary γ_i, and therefore also for $\gamma_i = \beta_i$, with β_i obtained by applying the improved resummation method to the exact S_l. With this choice the pure Coulomb resummed coefficients $\alpha_n^{C(r)}$ asymptotically approach $\alpha_n^{(r)}$ as rapidly as the pure Coulomb S-matrix elements, S_l^C, approach S_l in the usual optical potential calculations.

With the change of notation $f^{(0)} \equiv f$, $f^{(\mp 1)} \equiv f^{(\mp)}_{\{\beta\}}$, $f_R^{(0)} \equiv f_R$, $f_R^{(\mp 1)} \equiv f_{FR}^{(\mp)}$, $\mathcal{L}_n^{(0)} \equiv P_n$, and $\mathcal{L}_n^{(\mp 1)} \equiv Q_n^{(\mp)}$, by subtracting from (3), or (10), the corresponding resummed forms (12) applied to pure Coulomb scattering ($s = r, \gamma_i = \beta_i, \delta_n^{(s)} = \alpha_n^{C(r)}$ and $q = 0, \mp 1$), one obtains the final result

$$f^{(m)}(\theta) = \left(\prod_{i=0}^{r} \frac{1}{1+\beta_i x}\right) \sum_{n=0}^{\infty} [\alpha_n^{(r)} - \alpha_n^{C(r)}] \mathcal{L}_n^{(m)}(x) + f_R^{(m)}(\theta) + m \frac{i}{\pi} \sum_{i=0}^{r} \beta_i \alpha_0^{C(i-1)} \prod_{j=0}^{i} \frac{1}{1+\beta_j x}, \tag{13}$$

with $m = 0$ for the full amplitude and $m = \mp 1$ for the NF subamplitudes. For $r = 0$ and $m = 0$, or $m = \mp 1$, Eq. (13) is the usual regularization procedure defining the r.h.s of (3), or (10), in the presence of a long range Coulomb term in the potential. This procedure is based on adding the explicit expression of $f_R(\theta)$, or $f_{FR}^{(\mp)}(\theta)$, and subtracting its formal PWS, or LFS, for the full amplitude (Ref. [16], p. 428), or the Fuller NF subamplitudes [1]. For $r \geq 1$, Eq. (13) is the generalization of this regularization procedure to resummed forms of the full amplitude, or NF subamplitudes. The sum appearing in this term is as rapidly convergent as the usual sum with $r = 0$.

Before showing the effectiveness of our regularization procedure in a physically interesting case, we show the difficulties met by the EYRW technique [8] to ensure, and speed up, the convergence of improved, or not, LFS for pure Coulomb scattering. In this case $a_n \equiv a_n^C$, and the LFS on the r.h.s. of (13) is identically null, with arbitrary choice of β_i. For $r = 0$, Eq. (13) trivially states that the scattering amplitude ($m = 0$) is the Rutherford amplitude, and the NF subamplitudes ($m = \mp 1$) are the usual Fuller-Rutherford ones. For $r > 0$, by choosing β_i accordingly with the improved resummation method, Eq. (13) gives the explicit expression of the improved NF subamplitudes ($m = \mp 1$) in term of the usual Fuller-Rutherford ones, and of simple functions depending on β_i and $\alpha_0^{C(i-1)}$. For simplicity we will name *exact* this explicit expression for pure Coulomb improved NF subamplitudes.

In Fig. 1 the thick curves show the ratio to the Rutherford cross section, $\sigma_R(\theta)$, of the exact pure Coulomb improved F cross sections, of order $r = 0, 1$, and 2 ($r = 0$ meaning the original Fuller method). In the same figure the thin curves show the F cross sections obtained by forcing, and accelerating, the convergence of (10) with an additional EYRW resummation of order $s = 4$, and fixing the maximum number of the summed partial waves to $l_{\max} = 100$ and 1000. The results were obtained with $\eta = 10$, which is a typical value of the Sommerfeld parameter for heavy-ion scattering. For this η value the improved resummation parameters are $\beta_1 = 0.9802 + 0.1980i$ (for $r = 1$), $\beta_1 = 1.0072 + 0.1166i$ and $\beta_2 = 0.7804 + 0.6052i$ (for $r = 2$). Figure 1 shows that the final EYRW resummation [8] ensures the convergence of the LFS (10), but the convergence rate is low. For $\theta \lesssim 5°$ a numerically satisfactory result is not obtained even with $l_{\max} = 1000$. By fixing $l_{\max}$ and the final resummation order, the angle at with the truncated LFS disagrees with the exact result increases with the improved resummation order.

Figure 1 also shows that the improved resummation method reduces, particularly at forward angles, the unphysical F contribution present in the original Fuller NF method. However it does not suppress it, and is ineffective at $\theta \approx 180°$. This is an insurmountable difficulty connected with the NF splitting (7), mathematically continuing (at $\theta = 180°$) the N subamplitude into a F one, or vice versa. This also in absence of physically meaningful subamplitudes justifying this continuation. In these situations the only practical suggestion we can give is to not take seriously the NF subamplitudes at $\theta \approx 180°$, if in a neighbourood of this angle the cross section and the LIP of the full amplitude have a non oscillatory behavior, suggesting the dominance of a *single side* (positive LIP for F and negative for N) contribution. We remember that in [4] the LIP (local impact parameter) is defined as the derivative of the argument of the scattering amplitude with respect to the scattering angle, named LAM (local angular momentum) by Fuller[1], divided by the wavenumber k.

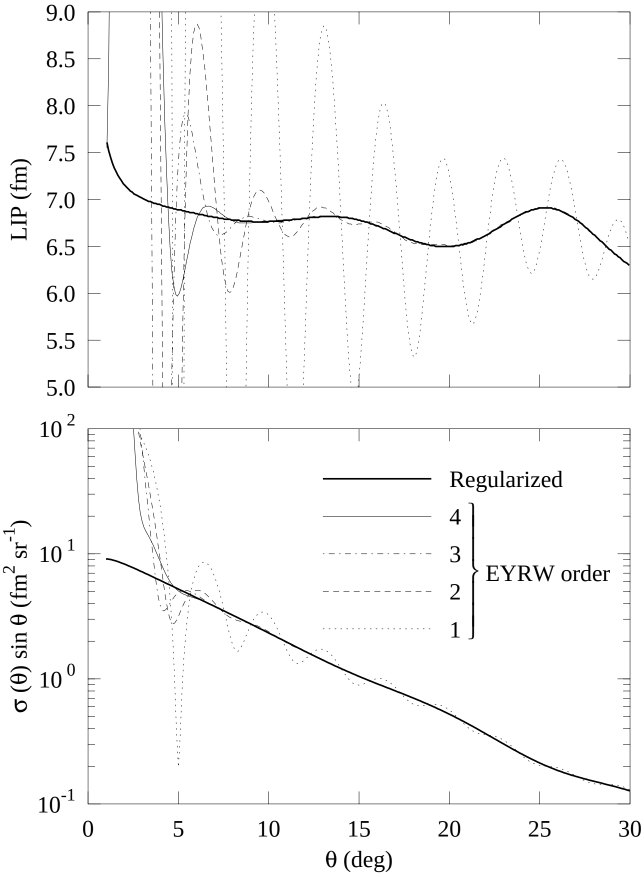

FIG. 2: First order ($r = 1$) improved F cross section (lower panel) and LIP (upper panel) for the $^{16}O + ^{16}O$ case. The calculations were done using 150 partial waves with our regularization procedure (thick curves) and with different final EYRW resummation orders (thin curves).

As a second example of the effectiveness of our regularization procedure, we consider the first order improved F cross section and LIP of the phenomenological optical potential WS2, used to fit [17] the $^{16}O + ^{16}O$ elastic cross section at $E_{\rm lab} = 145$ MeV. The improved resummation parameter is in this case $\beta_1 = -.9997 - 0.0798i$ [5]. The upper panel of Fig. 2 shows, for $\theta < 30°$ and $l_{\rm max} = 150$, the F LIP calculated using our regularization procedure (thick curve) and different order (thin lines) EYRW resummations [8]. The lower panel shows the corresponding F cross sections. Symmetrization effects were ignored.

Note that 150 partial waves are more than really necessary to obtain reliable scattering amplitudes using our regularization procedure. Using an EYRW resummation of order 1 (thin dotted curves), this partial wave number is not sufficient to obtain a satisfactory result. By increasing the EYRW resummation order it decreases the angular width of the region where the thin curves differ from the corresponding thick ones. However, for $\theta \lesssim 5°$, the 150 partial waves used are not enough, even using a fourth order final EYRW resummation.

These results show, in practical examples, that EYRW resummed LFS for asymptotically Coulombic S_l are convergent, with a convergence rate increasing by increasing the resummation order. Compared with the extension here given of the usual regularization procedure for asymptotically Coulombic S_l the EYRW resummation technique effectiveness is, however, computationally poor. The regularization procedure here described can be easily extended to make rapidly convergent the LFS in (3) and (10) for scattering by short range potentials. In these cases, however, also an additional first order EYRW resummation makes the LFS convergent with the same rapidity, and there is no practical advantage in using a different procedure.

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